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Scattering and Transmission Resonances in One-Dimensional Hulthén Potential Barrier

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Abstract The two component approach to the one-dimensional Dirac equation is applied to the Hulthén potential barrier. The scattering state solutions are obtained in terms of hypergeometric functions and the condition for a transmission resonance when the transmission coefficient is unity is derived.

Keywords Dirac equation · Hulthén potential barrier · Scattering state · Transmission resonance

1 Introduction

It is well known that the exact solutions of wave equation with certain physical potentials play an important role in quantum mechanics and both the bound, the scattering and transmission states of the quantum system in the framework of the Schrödinger equation have been known and understood well [1, 2]. The pioneering works on the computation of transmission and reflection coefficients in one-dimensional potential should go back to Bohm [3], who coined the definition of transmission resonance to scattering states with transmission coefficient equal to unity and vanishing reflection and showed that under certain conditions transmission resonance of the scattering states, characterized by transmission coefficient $T = 1$, and reflection coefficient $R = 0$, can be found for well-behaved potentials. After that, the transmission resonance has been carried out mainly in one-dimensional Schrödinger equation by Senn [4] and Sassoli de Bianchi [5]. Generally speaking, for the low momentum scattering by a physical potential, the transmission coefficient at zero energy become zero, but the reflection coefficient is unity, unless the potential supports a half-bound state. The so-called half-bound state is described by the wave function that is finite at infinity but is not square integrable [2].

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Recently, the study of the bound and scattering states of the one-dimensional quantum system have been generalized to the relativistic equations. The transmission resonance for a square barrier in the Dirac equation had been known [6] and their relationship to supercritical states had been pointed out [7]. For some symmetrical and well-behaved potentials, such as the Woods-Saxon one [8, 9], and the cusp one [10, 11], the transmission resonance of zero momentum may be observed, the corresponding potential well supports a half bound state at $E = -m$ as in the square potential case [12].

It is the purpose of this study to compute the scattering state solutions of one-dimensional Dirac equation in the presence of the spatially Hulthén potential barrier, and show the conditions for transmission resonance. The paper is organized as follows. In Sect. 2 the two component approach is used to establish the one-dimensional Dirac equation with an external potential $V(x)$. Section 3 is devoted to studying the one-dimensional Dirac equation with Hulthén potential barrier. The exact scattering state solutions are obtained. In Sect. 4 the transmission coefficient T and the reflection coefficient R are derived. The condition for the transmission resonance when transmission coefficient is unity is obtained from the conservation of charge. The concluding remarks are given in Sect. 5.

2 The One-Dimensional Dirac Equation

In the presence of an external spatially potential $V(x)$ and taking the gamma matrices γ_x and γ_0 to be the Pauli matrices $i\sigma_x$ and σ_z respectively, the one-dimensional Dirac equation can be written as ($\hbar = c = 1$) [8]:

$$\left(\sigma_x \frac{\mathrm{d}}{\mathrm{d}x} - (E - V(x))\sigma_z + m\right)\psi(x) = 0. \tag{1}$$

The Dirac spinor, $\psi(x)$, is decomposed in to two spinors u_1 and u_2, so that

$$\psi(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}. \tag{2}$$

Thus the problem of solving the one-dimensional Dirac equation (1) is reduced to that of finding the solutions to the coupled differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}x}u_1(x) = -(m + E - V(x))u_2(x), \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}u_2(x) = -(m - E + V(x))u_1(x). \tag{4}$$

Following a similar procedure to that used by Kennedy [8] and Flügge [13], introduce the following combinations

$$\phi(x) = u_1(x) + iu_2(x), \qquad \chi(x) = u_1(x) - iu_2(x). \tag{5}$$

Substituting these into (3) and (4), the re-arranging gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) = -im\chi(x) + i(E - V(x))\phi(x), \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\chi(x) = im\phi(x) - i(E - V(x))\chi(x). \tag{7}$$

The two components, $\phi(x)$ and $\chi(x)$, satisfy

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) + [(E - V(x))^2 - m^2 + iV'(x)]\phi(x) = 0, \tag{8}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\chi(x) + [(E - V(x))^2 - m^2 - iV'(x)]\chi(x) = 0. \tag{9}$$

Where the primes denote derivatives with respect to x. In the following the solutions for $\phi(x)$ will be presented by solving (8) analytically. The other component, $\chi(x)$, can be derived from (6).

3 The Hulthén Potential Barrier and Scattering States

The Hulthén potential is one of the important short range potentials [14] and has been applied to a number of areas such as nuclear and particle physics, atomic physics, condensed matter and chemical physics [15–18]. Within a non-relativistic framework the model of three-dimensional delta-function well could be considered as a Hulthén potential with the radius of the force going down to zero [19]. Moreover, the relativistic effect for a particle under the action of this potential could become important, especially for a strong coupling system [20]. The one-dimensional Hulthén potential barrier is defined as

$$V(x) = V_0\left[\Theta(x)\frac{\mathrm{e}^{-x/a}}{1 - \mathrm{e}^{-x/a}} + \Theta(-x)\frac{\mathrm{e}^{x/a}}{1 - \mathrm{e}^{x/a}}\right]. \tag{10}$$

Where the parameter V_0 defines the strength of the potential barrier and positive constant a determines the shape of potential, $\Theta(x)$ is the Heaviside function. The form of the Hulthén potential is shown in Fig. 1. It should be mentioned that Hulthén potential is not a smoothed-out form of the potential barrier and does not have a square barrier limit as the Woods-Saxon one, but this does not introduce any essential physical restriction on the problem.

First, we consider the scattering solutions for $x < 0$ with $|E| > m$. In this case and on making the substitution $y = \mathrm{e}^{-x/a}$, (8) becomes

$$y^2\frac{\mathrm{d}^2\phi_L(y)}{\mathrm{d}y^2} + y\frac{\mathrm{d}\phi_L(y)}{\mathrm{d}y} + a^2\left[\left(E + \frac{V_0}{1-y}\right)^2 - m^2 + iV_0\frac{V_0 y}{a(1-y)^2}\right]\phi_L(y) = 0. \tag{11}$$

Splitting off powers of y and $(1-y)$ by setting $\phi_L(y) = y^{\mu}(1-y)^{-\lambda}f(y)$ and substituting it into the above equation reduces it to the hypergeometric equation

$$y(1-y)\frac{\mathrm{d}^2}{\mathrm{d}y^2}f(y) + [(1+2\mu) - (1-2\mu-2\lambda)y]\frac{\mathrm{d}}{\mathrm{d}y}f(y) - (\mu-\lambda-\nu)(\mu+\lambda-\nu)f(y) = 0, \tag{12}$$

where

$$\begin{aligned} &\mu = iap, \qquad p^2 = (E^2 - V_0^2) - m^2, \\ &\nu = iak, \qquad k^2 = (E^2 - m^2), \qquad \lambda = -iaV_0. \end{aligned} \tag{13}$$

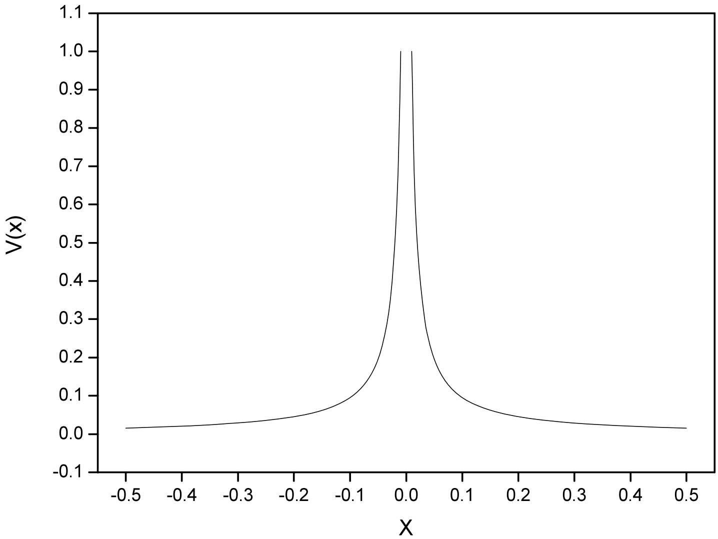

Fig. 1 The Hulthén potential barrier for $a = 1$, $x = \pm 0.5 \sim \pm 0.01$

Equation (12) has the general solution

$$f(y) = F(\mu - \nu - \lambda, \mu + \nu - \lambda, 1 + 2\mu, y). \tag{14}$$

So

$$\phi_L(y) = y^{\mu}(1-y)^{-\lambda} F(\mu - \nu - \lambda, \mu + \nu - \lambda, 1 + 2\mu, y). \tag{15}$$

For this to be a physically acceptable solution to the problem, it must satisfy the appropriate boundary conditions as $x \to -\infty$. The solutions as $x \to -\infty \Rightarrow y \to \infty$ can be determined by using the following asymptotic behaviour of the hypergeometric function [13]

$$F(y) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-y)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-y)^{-b}. \tag{16}$$

And noting that in the limiting $x \to -\infty$, $(y)^{\pm\nu} = \mathrm{e}^{\mp ikx}$, $(1-y)^{-\lambda} = (-y)^{-\lambda}$. In this limit, $\phi_L(x)$ can be written as

$$\lim_{x\to\infty} \phi_L(x) = A\mathrm{e}^{ikx} + B\mathrm{e}^{-ikx}. \tag{17}$$

From (6) the other component, $\chi(x)$, is

$$\chi(x) = \frac{1}{im}[i(E - V(x))\phi(x) - \phi'(x)]. \tag{18}$$

Substituting (17) into the above gives

$$\lim_{x\to\infty} \chi_L(x) = A\left(\frac{E-k}{m}\right)\mathrm{e}^{ikx} + B\left(\frac{E+k}{m}\right)\mathrm{e}^{-ikx}. \tag{19}$$

Where the condition $V(x) \to 0$ as $x \to -\infty$ is considered. The Dirac spinors, $u_1(x)$ and $u_2(x)$, can be written as

$$u_1(x) = \frac{1}{2}\left[A\left(\frac{m+E-k}{m}\right)e^{ikx} + B\left(\frac{m+E+k}{m}\right)e^{-ikx}\right], \tag{20}$$

$$u_2(x) = \frac{1}{2}\left[A\left(\frac{m-E+k}{m}\right)e^{ikx} + B\left(\frac{m-E-k}{m}\right)e^{-ikx}\right]. \tag{21}$$

Where in the above

$$A = \frac{\Gamma(1+2\mu)\Gamma(-2\nu)}{\Gamma(\mu-\nu-\lambda)\Gamma(1+\mu-\nu+\lambda)}e^{(ka-i\mu)\pi}, \tag{22}$$

$$B = \frac{\Gamma(1+2\mu)\Gamma(2\nu)}{\Gamma(\mu+\nu-\lambda)\Gamma(1+\mu+\nu+\lambda)}e^{-(ka+i\mu)\pi}. \tag{23}$$

It can be seen from the above that the wave function, $\psi(x)$, comprises of an incident and reflected wave far to the left of the barrier which is a desired form to establish reflected and transmission coefficients.

Now we consider the solutions for $x > 0$. In this case choosing $z = e^{-x/a}$ and substitute it into (8) yields

$$z^2\frac{d^2}{dz^2}\phi_R(z) + z\frac{d}{dz}\phi_R(z) + a^2\left[\left(E - V_0\frac{z}{1-z}\right)^2 - m^2 - i\frac{V_0 z}{a(1-z)^2}\right]\phi_R(z) = 0. \tag{24}$$

Putting $\phi_R(z) = z^{-\nu}(1-z)^{\lambda}g(z)$ and substituting it into (24) reduce it to the following hypergeometric equation

$$z(1-z)\frac{d^2}{dz^2}g(z) + [(1-2\lambda)-(1-2\nu+2\lambda)z]\frac{d}{dz}g(z) - (-\nu+\lambda+\mu)(-\nu+\lambda-\mu)g(z) = 0. \tag{25}$$

Equation (25) has the general solution

$$\begin{aligned} g(z) &= d_1 F(-\nu+\lambda+\mu, -\nu+\lambda-\mu, 1-2\nu, z) \\ &\quad + d_2 z^{2\nu} F(\nu+\lambda+\mu-1, \nu+\lambda-\mu-1, 1+2\nu, z). \end{aligned} \tag{26}$$

So

$$\begin{aligned} \phi_R(z) &= d_1 z^{-\nu}(1-z)^{\lambda} F(-\nu+\lambda-\mu, -\nu+\lambda+\mu, 1-2\nu, z) \\ &\quad + d_2 z^{\nu}(1-z)^{\lambda} F(\nu+\lambda+\mu-1, \nu+\lambda-\mu-1, 1+2\nu, z). \end{aligned} \tag{27}$$

Also as $x \to \infty$, $z \to 0$, $(1-z) = 1$, and $z^{-\nu} = e^{ikx}$. Therefore in order to have a plane wave travelling to the right as $x \to \infty$, $d_2 = 0$. So

$$\phi_R(z) = d_1 z^{-\nu}(1-z)^{\lambda} F(-\nu+\lambda-\mu, -\nu+\lambda+\mu, 1-2\nu, z), \tag{28}$$

and

$$\lim_{x\to\infty} \phi_R(x) = d_1 e^{ikx}. \tag{29}$$

Whilst the other component

$$\lim_{x\to\infty}\chi_R(x)=d_1\left(\frac{E-k}{m}\right)e^{i\pi\nu}e^{ikx}. \tag{30}$$

From (5), the Dirac spinors, $u_1(x)$ and $u_2(x)$, can be derived as

$$u_1(x)=\frac{1}{2}d_1\left(\frac{m+E-k}{m}\right)e^{ikx}, \tag{31}$$

$$u_2(x)=-\frac{i}{2}d_1\left(\frac{m-E+k}{m}\right)e^{ikx}. \tag{32}$$

4 Transmission Resonance Condition

In order to find out the condition of transmission resonance, we begin with a discussion of the current density. The electrical current density for one-dimensional Dirac equation (1) is defined as [8]

$$j=\bar{\psi}(x)\gamma_x\psi(x)=\frac{1}{2}(|\phi(x)|^2-|\chi(x)|^2). \tag{33}$$

The current density as $x\to-\infty$ is $j_L=j_{in}-j_{refl}$, where j_{in} is the incident current density and j_{refl} is the reflected one. Similarly as $x\to\infty$, we have that the current density $j_R=j_{trans}$ where j_{trans} is the transmitted current. Substituting (17), (19), (29) and (30) into (33) we find that

$$j_L=|A|^2\frac{k}{m^2}(E-k)-|B|^2\frac{k}{m^2}(E+k), \tag{34}$$

$$j_R=|d_1|^2\frac{k}{m^2}(E-k). \tag{35}$$

From the conservation of charge we have that $j_L=j_R$ which together with the reflection coefficient, R, and the transmission coefficient, T

$$R=\frac{|B|^2}{|A|^2}\left(\frac{E+k}{E-K}\right),\qquad T=\frac{|d_1|^2}{|A|^2}, \tag{36}$$

we obtained the unitarity condition

$$T+R=1. \tag{37}$$

From (37) we find that transmission coefficient T and reflection coefficient R satisfy the relation $T=1-R$. This gives

$$T=1-\frac{|B|^2}{|A|^2}\left(\frac{E+k}{E-k}\right). \tag{38}$$

From (22) and (23) we get

$$\frac{B}{A}=\frac{\Gamma(1+2\mu)\Gamma(2\nu)\Gamma(\mu-\nu-\lambda)\Gamma(1+\mu-\nu+\lambda)e^{-(ka+i\mu)\pi}}{\Gamma(1+2\mu)\Gamma(-2\nu)\Gamma(\mu+\nu-\lambda)\Gamma(1+\mu+\nu+\lambda)e^{(ka-i\mu)\pi}}. \tag{39}$$

Since $\nu = iak$ is imaginary, the ratio of $\Gamma(2\nu)$ to $\Gamma(-2\nu)$ has no contribution to $|B|^2/|A|^2$. Equation (39) is reduced as

$$\frac{B}{A} = \frac{(\mu-\nu+\lambda)\Gamma(\mu-\nu+\lambda)\Gamma(\mu-\nu-\lambda)}{(\mu+\nu+\lambda)\Gamma(\mu+\nu+\lambda)\Gamma(\mu+\nu-\lambda)} e^{-2ka}. \tag{40}$$

Where we used the relation $\Gamma(z+1) = z\Gamma(z)$. Substituting (13) into (38), we obtained

$$T = 1 - \frac{|(\mu-\nu+\lambda)\Gamma(\mu-\nu+\lambda)\Gamma(\mu-\nu-\lambda)|^2}{|(\mu+\nu+\lambda)\Gamma(\mu+\nu+\lambda)\Gamma(\mu+\nu-\lambda)|^2} e^{-4ka\pi} \left(\frac{E+k}{E-k}\right)^2. \tag{41}$$

From (41) we obtain that the transmission resonance condition when the transmission coefficient is unity satisfy the equation,

$$|(\mu-\nu+\lambda)\Gamma(\mu-\nu+\lambda)\Gamma(\mu-\nu-\lambda)| e^{-2ka\pi} = 0. \tag{42}$$

5 Conclusions

In this study the one-dimensional Dirac equation with the Hulthén potential barrier, which does not have a square barrier limit as the Woods-Saxon one is solved. The exact scattering state solutions are obtained in terms of hypergeometric functions. It is shown that the Dirac spinor comprised two components are satisfied the appropriate boundary conditions as $x \to -\infty$ and $x \to \infty$. The condition for the transmission resonance is obtained from the conservation of charge.

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